AMC Math Competition Questions about Divisibility

(2015 – 2021)

MORE VIDEOS

This document is a summary of AMC questions about divisibility. The answers and explanations are in our Youtube Channel.

https://eureka314.com/
https://www.youtube.com/channel/UCRbhZCQK9IqiH7ZSaXQidpA/videos

Question Distribution:

Year	Easy (1-12)	Intermediate (13-20)	Hard (21-25)	Total
2021 10B	12			1
2020 10A	6,	15,	22, 24	4
2020 10B	7,12	14,	22,	4
2019 10A	9,11	15	25	4
2019 10B		14,19		2
2018 10A	7	17	22	3
2018 10B	11	14,16	21,23,25	6
2017 10A		16		1
2017 10B		14	23	2
2016 10A			22,25	2
2015 10B			23	1
2014 10B	12	17		2
2013 10A		13		1
Total	9	12	12	33

Commonly Used Knowledge

Divisibility Rules

- Divisible by 2: All even numbers (Numbers with unit digit as 0, 2, 4, 6, 8).
- Divisible by 3: Sum of the digits divisible by 3. Example: 111 as 1 + 1 + 1 is divisible by 3.
- Divisible by 4: Last two digits divisible by 4. Example: 1024.
- Divisible by 5: Numbers with unit digit as 0 or 5.
- Divisible by 6: Divisible by 2 and 3.
- Divisible by 8: Last three digits divisible by 8. Example: 2048.
- Divisible by 9: Sum of the digits divisible by 9.
- Divisible by 10: Numbers with unit digit as 0.
- Divisible by 7:
    - From right to left, divide the number to groups of 3 digits.
    - From right to left, subtract even number of groups from odd number of groups.
    - Sum all the results.
    - If the result divisible by 7, the number is divisible by 7.
    - There are other rules for divisible by 7. Find the one easier for you to remember.
    - Example: 1234567
        - First, we have 1, 234, 567
        - Then we get 567 - 234 + 1 = 334 = 47 X 7 + 5, so it's not divisible by 7
    - Another example: 1234569
        - First, we have 1, 234, 569
        - Then we get 569 - 234 + 1 = 336 = 48 X 7, so it's divisible by 7
- Divisible by 11
    - Sum all the digit in even places.
    - Sum all the digits in odd places.
    - Find the difference between the above two numbers.
    - The result must be divisible by 11.
    - Example: 789,789
    - Sum of even digits = 7 + 9 + 8 = 24
    - Sum of odd digits = 8 + 7 + 9 = 24
    - The difference is 0 (multiple of 11), so the number is divisible by 11.
- To test divisibility by a composite number n (doesn't prove anything when n is prime), you find 2 numbers, d_1 and d_2, such as LCM($d_1,\ d_2$) is n. Then, n is divisible by the number if and only if the number is divisible by d_1 and d_2.

Number of Divisors

If a number can be written as a product of prime factors: n = p ^{a} q ^{b} r ^{c} ... then the number of divisors, **d(n) = (a +1)(b +1)(c +1)**...

https://eureka314.com/
https://www.youtube.com/channel/UCRbhZCQK9IqiH7ZSaXQidpA/videos

12 = 2^2*3^1

12 has (2+1)(1+1) = 6 divisors. They are 1, 2, 3, 4, 6, 12.

Legendre's formula

The exponent of the largest power of *p* that divides *n!*

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$$

Or

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1}$$

Where $s_p(n)$ is the sum of digits of n in base p.
Example:

$$\nu_2(6!) = \sum_{i=1}^{\infty} \left\lfloor \frac{6}{2^i} \right\rfloor = \left\lfloor \frac{6}{2} \right\rfloor + \left\lfloor \frac{6}{4} \right\rfloor = 3 + 1 = 4,$$

$$\nu_3(6!) = \sum_{i=1}^{\infty} \left\lfloor \frac{6}{3^i} \right\rfloor = \left\lfloor \frac{6}{3} \right\rfloor = 2,$$

$$\nu_5(6!) = \sum_{i=1}^{\infty} \left\lfloor \frac{6}{5^i} \right\rfloor = \left\lfloor \frac{6}{5} \right\rfloor = 1.$$

Example:
6 is 110 in base 2, 20 in base 3 and 11 in base 5. We can get the same result as the previous slide.

Properties of LCM and GCD

The least common multiple (LCM) of two integers is the **smallest positive integer** that is a multiple of both. The greatest common divisor (GCD) of two integers is the largest positive integer dividing both.

The product of the two numbers is the product of the LCM and the GCD.
If GCD(a, b) = m, LCM(a, b) = n, then ab = mn.
Also:
a | n, b | n, m | a, m | b

https://eureka314.com/
https://www.youtube.com/channel/UCRbhZCQK9IqiH7ZSaXQidpA/videos

Euclidean Algorithm

Euclidean Algorithm is a way to get the greatest common divisor (GCD). The basic format of Euclidean Algorithm is:

gcd(a, b)
= gcd(a - b, b)
= gcd(a, b - a)
We can also use the Euclidean Algorithm the reversed way:
gcd(a, b)
= gcd(a + b, b)
= gcd(a, b + a)

Example:
gcd(99, 81)
= gcd(99 - 81, 81)
= gcd(18, 81)
= gcd(18, 81 – 18 * 4)
= gcd(18, 9)
= gcd (9, 9)
= 9

Fermat's Little Theorem

if p is a prime number, then for any integer a

$$a^p \equiv a \pmod{p}.$$

Example:
Let a = 5, p = 7, 5^7 **=** 78125 = 11160 * 7 + 5
So: $5^7 \equiv 5 \pmod{7}$

If a is not divisible by p, we also have

$$a^{p-1} \equiv 1 \pmod{p}.$$

Example:

$8^{(5-1)} \equiv 1 \pmod{5}$

https://eureka314.com/
https://www.youtube.com/channel/UCRbhZCQK9IqiH7ZSaXQidpA/videos

Chinese Remainder Theorem

The Chinese remainder theorem asserts that if the ni are pairwise coprime, and if $a_1, \ldots, a_k$ are integers such that $0 \le a_i < n_i$ for every i, then there is one and only one integer x, such that $0 \le x < N$ and the remainder of the Euclidean division of x by n_i is a_i for every i.

$$(S):\quad \begin{cases} x \equiv a_1 \pmod{m_1} \\ x \equiv a_2 \pmod{m_2} \\ \quad\vdots \\ x \equiv a_n \pmod{m_n} \end{cases}$$

$$M = m_1 \times m_2 \times \cdots \times m_n = \prod_{i=1}^{n} m_i:$$

$$M_i = M/m_i, \quad \forall i \in \{1, 2, \cdots, n\}$$

$$t_i M_i \equiv 1 \pmod{m_i}, \quad \forall i \in \{1, 2, \cdots, n\}.$$

$$x = a_1 t_1 M_1 + a_2 t_2 M_2 + \cdots + a_n t_n M_n + kM = kM + \sum_{i=1}^{n} a_i t_i M_i$$

$$x = \sum_{i=1}^{n} a_i t_i M_i \qquad x = a_1 t_1 M_1 + a_2 t_2 M_2 + \cdots + a_n t_n M_n + kM = kM + \sum_{i=1}^{n} a_i t_i M_i$$

Example from in the Sun-tzu Suan-ching in the 3rd century CE:
A numb $x = \sum_{i=1}^{n} a_i t_i M_i$ nainder 2 if divided by 3, remainder 3 if divided by 5, and reminder 2 if divided by 7.

First, we know:
m1 = 3, m2 = 5, m3 = 7. M = 3 * 5 * 7 = 105, and
M1 = 35, M2 = 21, M3 = 15
For the multiple of 35, 70 divide 3 has remainder 1. So t1 = 2 (70/35). Similarly, we can get t2 = 1 and t3 = 1

The number would be $t_1M_1 + t_2M_2 + t_3M_3 = 233$. x can be any number which
x = 233 + k * 105.
The smallest positive x is 23 (when k = -2)

Polynomial long division

We know how to divide numbers. It might look strange to divide polynomials but it's possible. First, we order the items from biggest power to the smallest. If there is a missing power, we add 0 times that power instead. Then we can use the same way as long division.

https://eureka314.com/
https://www.youtube.com/channel/UCRbhZCQK9IqiH7ZSaXQidpA/videos

$$
\begin{array}{r}
x - 10 \\
x^2 - 2x + 1 \overline{) x^3 - 12x^2 + 0x - 42} \\
\underline{x^3 - \ \ 2x^2 + \ \ x } \\
-10x^2 - \ \ \ x - 42 \\
\underline{-10x^2 + 20x - 10} \\
-21x - 32
\end{array}
$$

Sophie Germain Identity

$$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$$

Recent Problems (2015 – 2021)

2021 AMC 10B Problem 12

Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

$\textbf{(A)}\ 1:16 \qquad \textbf{(B)}\ 1:15 \qquad \textbf{(C)}\ 1:14 \qquad \textbf{(D)}\ 1:8 \qquad \textbf{(E)}\ 1:3$

2020 AMC 10A Problem 6

How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by $5?$

$\textbf{(A)}\ 80 \qquad \textbf{(B)}\ 100 \qquad \textbf{(C)}\ 125 \qquad \textbf{(D)}\ 200 \qquad \textbf{(E)}\ 500$

2020 AMC 10A Problem 15

A positive integer divisor of $12!$ is chosen at random. The probability that the divisor chosen is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

$\textbf{(A) } 3 \qquad \textbf{(B) } 5 \qquad \textbf{(C) } 12 \qquad \textbf{(D) } 18 \qquad \textbf{(E) } 23$

2020 AMC 10A Problem 22

For how many positive integers $n \le 1000$ is $\left\lfloor \dfrac{998}{n} \right\rfloor + \left\lfloor \dfrac{999}{n} \right\rfloor + \left\lfloor \dfrac{1000}{n} \right\rfloor$ not divisible by 3? (Recall that $\lfloor x \rfloor$ is the greatest integer less than or equal to x.)

$\textbf{(A) } 22 \qquad \textbf{(B) } 23 \qquad \textbf{(C) } 24 \qquad \textbf{(D) } 25 \qquad \textbf{(E) } 26$

2020 AMC 10A Problem 24

Let n be the least positive integer greater than 1000 for which $\gcd(63, n+120) = 21 \quad \text{and} \quad \gcd(n+63, 120) = 60.$ What is the sum of the digits of n?

$\textbf{(A) } 12 \qquad \textbf{(B) } 15 \qquad \textbf{(C) } 18 \qquad \textbf{(D) } 21 \qquad \textbf{(E) } 24$

2020 AMC 10B Problem 7

How many positive even multiples of 3 less than 2020 are perfect squares?

$\textbf{(A) } 7 \qquad \textbf{(B) } 8 \qquad \textbf{(C) } 9 \qquad \textbf{(D) } 10 \qquad \textbf{(E) } 12$

2020 AMC 10B Problem 12

The decimal representation of $\frac{1}{20^{20}}$ consists of a string of zeros after the decimal point, followed by a 9 and then several more digits. How many zeros are in that initial string of zeros after the decimal point?

$\textbf{(A) } 23 \qquad \textbf{(B) } 24 \qquad \textbf{(C) } 25 \qquad \textbf{(D) } 26 \qquad \textbf{(E) } 27$

2020 AMC 10B Problem 22

What is the remainder when $2^{202}+202$ is divided by $2^{101}+2^{51}+1$?

$\textbf{(A) } 100 \qquad\textbf{(B) } 101 \qquad\textbf{(C) } 200 \qquad\textbf{(D) } 201 \qquad\textbf{(E) } 202$

2019 AMC 10A Problem 9

What is the greatest three-digit positive integer n for which the sum of the first n positive integers is $\underline{not}$ a divisor of the product of the first n positive integers?

$\textbf{(A) } 995 \qquad\textbf{(B) } 996 \qquad\textbf{(C) } 997 \qquad\textbf{(D) } 998 \qquad\textbf{(E) } 999$

2019 AMC 10A Problem 11

How many positive integer divisors of 201^9 are perfect squares or perfect cubes (or both)?

$\textbf{(A) } 32 \qquad\textbf{(B) } 36 \qquad\textbf{(C) } 37 \qquad\textbf{(D) } 39 \qquad\textbf{(E) } 41$

2020 AMC 10A Problem 15

A positive integer divisor of $12!$ is chosen at random. The probability that the divisor chosen is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

$\textbf{(A) } 3 \qquad\textbf{(B) } 5 \qquad\textbf{(C) } 12 \qquad\textbf{(D) } 18 \qquad\textbf{(E) } 23$

2019 AMC 10A Problem 25

For how many integers n between 1 and 50, inclusive, is $\frac{(n^2-1)!}{(n!)^n}$ an integer? (Recall that $0! = 1$.)

$\textbf{(A) } 31 \qquad\textbf{(B) } 32 \qquad\textbf{(C) } 33 \qquad\textbf{(D) } 34 \qquad\textbf{(E) } 35$

2019 AMC 10B Problem 14

The base-ten representation for $19!$ is $121,6T5,100,40M,832,H00$, where T, M, and H denote digits that are not given. What is $T+M+H$?

$\textbf{(A) }3 \qquad\textbf{(B) }8 \qquad\textbf{(C) }12 \qquad\textbf{(D) }14 \qquad\textbf{(E) }17$

2019 AMC 10B Problem 19

Let S be the set of all positive integer divisors of $100,000.$ How many numbers are the product of two distinct elements of S?

$\textbf{(A) }98\qquad\textbf{(B) }100\qquad\textbf{(C) }117\qquad\textbf{(D) }119\qquad\textbf{(E) }121$

2018 AMC 10A Problem 7

For how many (not necessarily positive) integer values of n is the value of $4000\cdot \left(\tfrac{2}{5}\right)^n$ an integer?

$\textbf{(A) }3 \qquad \textbf{(B) }4 \qquad \textbf{(C) }6 \qquad \textbf{(D) }8 \qquad \textbf{(E) }9$

2018 AMC 10A Problem 17

Let S be a set of 6 integers taken from $\{1,2,\dots,12\}$ with the property that if a and b are elements of S with $a<b$, then b is not a multiple of a. What is the least possible value of an element in S?

$\textbf{(A)}\ 2\qquad\textbf{(B)}\ 3\qquad\textbf{(C)}\ 4\qquad\textbf{(D)}\ 5\qquad\textbf{(E)}\ 7$

2018 AMC 10A Problem 22

Let $a, b, c,$ and d be positive integers such that $\gcd(a, b)=24$, $\gcd(b, c)=36$, $\gcd(c, d)=54$, and $70<\gcd(d, a)<100$. Which of the following must be a divisor of a?

$\textbf{(A)} \text{ 5} \qquad \textbf{(B)} \text{ 7} \qquad \textbf{(C)} \text{ 11} \qquad \textbf{(D)} \text{ 13} \qquad \textbf{(E)} \text{ 17}$

2018 AMC 10B Problem 11

Which of the following expressions is never a prime number when p is a prime number?

(A) p^2+16 **(B)** p^2+24 **(C)** p^2+26 **(D)** p^2+46 **(E)** p^2+96

2018 AMC 10B Problem 13

How many of the first 2018 numbers in the sequence $101, 1001, 10001, 100001, \dots$ are divisible by 101?

(A) 253 **(B)** 504 **(C)** 505 **(D)** 506 **(E)** 1009

2018 AMC 10B Problem 16

Let $a_1, a_2, \dots, a_{2018}$ be a strictly increasing sequence of positive integers such that $a_1 + a_2 + \cdots + a_{2018} = 2018^{2018}$. What is the remainder when $a_1^3 + a_2^3 + \cdots + a_{2018}^3$ is divided by 6?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

2018 AMC 10B Problem 21

Mary chose an even 4-digit number n. She wrote down all the divisors of n in increasing order from left to right: $1, 2, \dots, \dfrac{n}{2}, n$. At some moment Mary wrote 323 as a divisor of n. What is the smallest possible value of the next divisor written to the right of 323?

(A) 324 **(B)** 330 **(C)** 340 **(D)** 361 **(E)** 646

2018 AMC 10B Problem 23

How many ordered pairs (a, b) of positive integers satisfy the equation $a \cdot b + 63 = 20 \cdot \text{lcm}(a, b) + 12 \cdot \text{gcd}(a, b)$, where $\text{gcd}(a, b)$ denotes the greatest common divisor of a and b, and $\text{lcm}(a, b)$ denotes their least common multiple?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

2018 AMC 10B Problem 25

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many real numbers x satisfy the equation $x^2 + 10,000\lfloor x \rfloor = 10,000x$?

$\textbf{(A) } 197 \qquad \textbf{(B) } 198 \qquad \textbf{(C) } 199 \qquad \textbf{(D) } 200 \qquad \textbf{(E) } 201$

2017 AMC 10A Problem 16

There are 10 horses, named Horse 1, Horse 2, . . ., Horse 10. They get their names from how many minutes it takes them to run one lap around a circular race track: Horse k runs one lap in exactly k minutes. At time 0 all the horses are together at the starting point on the track. The horses start running in the same direction, and they keep running around the circular track at their constant speeds. The least time $S > 0$, in minutes, at which all 10 horses will again simultaneously be at the starting point is $S = 2520$. Let $T > 0$ be the least time, in minutes, such that at least 5 of the horses are again at the starting point. What is the sum of the digits of T?

$\textbf{(A) } 2 \qquad \textbf{(B) } 3 \qquad \textbf{(C) } 4 \qquad \textbf{(D) } 5 \qquad \textbf{(E) } 6$

2017 AMC 10B Problem 14

An integer N is selected at random in the range $1 \leq N \leq 2020$. What is the probability that the remainder when N^{16} is divided by 5 is 1?

$\textbf{(A) } \frac{1}{5} \qquad \textbf{(B) } \frac{2}{5} \qquad \textbf{(C) } \frac{3}{5} \qquad \textbf{(D) } \frac{4}{5} \qquad \textbf{(E) } 1$

2017 AMC 10B Problem 23

Let $N = 123456789101112\dots4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

$\textbf{(A) } 1 \qquad \textbf{(B) } 4 \qquad \textbf{(C) } 9 \qquad \textbf{(D) } 18 \qquad \textbf{(E) } 44$

2016 AMC 10A Problem 22

For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

$\textbf{(A) } 110 \qquad \textbf{(B) } 191 \qquad \textbf{(C) } 261 \qquad \textbf{(D) } 325 \qquad \textbf{(E) } 425$

2016 AMC 10A Problem 25

How many ordered triples (x,y,z) of positive integers satisfy $\text{lcm}(x,y) = 72, \text{lcm}(x,z) = 600$ and $\text{lcm}(y,z) = 900$?

$\textbf{(A) } 15 \qquad \textbf{(B) } 16 \qquad \textbf{(C) } 24 \qquad \textbf{(D) } 27 \qquad \textbf{(E) } 64$

2015 AMC 10A Problem 25

Let S be a square of side length 1. Two points are chosen at random on the sides of S. The probability that the straight-line distance between the points is at least $\frac{1}{2}$ is $\frac{a-b\pi}{c}$, where a, b, and c are positive integers with $\gcd(a,b,c)=1$. What is $a+b+c$?

$\textbf{(A) } 59 \qquad \textbf{(B) } 60 \qquad \textbf{(C) } 61 \qquad \textbf{(D) } 62 \qquad \textbf{(E) } 63$

2015 AMC 10B Problem 23

Let n be a positive integer greater than 4 such that the decimal representation of $n!$ ends in k zeros and the decimal representation of $(2n)!$ ends in $3k$ zeros. Let s denote the sum of the four least possible values of n. What is the sum of the digits of s?

$\textbf{(A) } 7 \qquad \textbf{(B) } 8 \qquad \textbf{(C) } 9 \qquad \textbf{(D) } 10 \qquad \textbf{(E) } 11$

2014 AMC 10B Problem 12

The largest divisor of $2,014,000,000$ is itself. What is its fifth largest divisor?

$\textbf{(A) } 125,875,000 \qquad \textbf{(B) } 201,400,000 \qquad \textbf{(C) } 251,750,000 \qquad \textbf{(D) } 402,800,000 \qquad \textbf{(E) } 503,500,000$

2014 AMC 10B Problem 17

What is the greatest power of 2 that is a factor of $10^{1002} - 4^{501}$?

(A) 2^{1002} **(B)** 2^{1003} **(C)** 2^{1004} **(D)** 2^{1005} **(E)** 2^{1006}

2013 AMC 10A Problem 13

How many three-digit numbers are not divisible by 5, have digits that sum to less than 20, and have the first digit equal to the third digit?

(A) 52 **(B)** 60 **(C)** 66 **(D)** 68 **(E)** 70